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Trilocal Structures. I. Secular Equation

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The principles used with the bilocal photon are applied to the analysis of trilocal structures. From the trilocal wave equation, and the phase-space boundary condition (that the structure float on a Fermi sea filling the vacuum), a secular equation is obtained, an eighth-order polynomial in the energy of the structure. Requiring that the structure move as a particle, with $w^2 = k^2 + m^2$, provides explicit expressions for two auxiliary parameters, λ and μ, to be used later in an expansion of the wave function.

1. INTRODUCTION

The basic concepts to be used here have been discussed in some detail in a previous article (Clapp, 1980). It is postulated that there exists a single primitive field, ξ, from which all particles and observable fields are to be built. The properties of ξ are to be inferred from its singleness.

The primitive field ξ is a massless fermion field with a σ spin of one-half. The field also has a τ spin of one-half, to specify either right-handed ($\tau_\xi = +1$) or left-handed ($\tau_\xi = -1$) helicity. Thus there are four spin possibilities for a primitive quantum. In general there can be as many as four quanta at one point in space, the necessary antisymmetrization being achieved via the two kinds of spin.

In the previous article, it was found that the vacuum needs to be a Fermi sea, but a Fermi sea of a particular character, in which the openness of the sea could be preserved only by the requirement that a quantum in the sea describe a pair of waves, not just one wave. In the vacuum sea, therefore, there can, at one point in space, be at most two primitive quanta, not four.

Structures representing elementary particles are standing-wave systems floating on the vacuum sea. The outer portions of a structure must resolve into a combination of incoming and outgoing primitive waves

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whose character permits this combination to merge into the surface of the vacuum sea. As long as this phase-space boundary condition is met, the inner details of the structure can be very complicated and involve many primitive quanta.

The previous article examined a particular structure, a two-quantum structure in which the two structural quanta had opposite τ spin. This permitted the inclusion in the wave function of terms in which the two σ spins were parallel. The structure examined had a total internal angular momentum of $J=1$, and moved as a massless particle at the velocity c. Parameters were chosen so that there were only two polarizations (rather than three). It was possible to define field quantities that resembled electric and magnetic field vectors and satisfied the vacuum Maxwell equations. This two-quantum structure is thus a reasonable candidate for a bilocal photon.

From a computationally pragmatic point of view, the structure that comes next is a trilocal structure with $J=1/2$. At the center of this structure, we can have all three quanta occupying the same point in space, but antisymmetry then requires that two of the three quanta have opposite τ spin and parallel σ spin. Such a pair of quanta, as part of a larger structure, could play the role of an internalized attached photon, contributing the electric and magnetic fields that accompany a charged particle such as the observed electron and muon.

A structure representing a charged lepton should move through space satisfying the relativistic equation which relates its energy, mass, and momentum. As will be seen, this requirement places restrictions on a trilocal structure, and by extension on structures formed from more than three primitive quanta.

As applied to the trilocal system, the wave equations (one centroid-time equation and two relative-time equations) and the boundary condition lead to matrix equations of infinite size, but the imposition of auxiliary conditions, obtained through the use of operators commuting with the wave equations, reduces the infinite matrix equations to finite size. The restrictions obtained from the secular equation then limit the eigenvalues of those auxiliary operators.

There are three wave equations for the trilocal system, the familiar centroid-time wave equation and two unfamiliar relative-time equations. The trilocal wave function must satisfy all three. By extension, an N-quantum structure will have N wave equations for the N-local wave function to satisfy, and this wave function will contain a functional dependence upon $(N-1)$ relative-time variables. This dependence is necessary if the wave function is to be fully relativistic.

Among the operators commuting with the wave equations are helicity operators, products of the angular momentum $\mathbf{J}$, and the internal and external momentum vectors. These will be found to aid in sorting the trilocal solutions, and by extension the N-local solutions.

When a complete set of commuting operators has been assembled, and a wave function is required to be an eigenfunction of all of the operators in the set, the result is the limitation of the eigenvalues of those operators to certain discrete sets, and in particular the limitation of the rest mass to certain explicit magnitudes. These are then the particle masses obtained from the theory.

2. TRILOCAL WAVE EQUATIONS

We will require that the trilocal wave function, $\Psi(1,2,3)$, should satisfy the three (truncated) wave equations

$$0=\left[\frac{1}{ic}\frac{\partial}{\partial t_1}+\frac{1}{i}\tau_{1\zeta}\boldsymbol{\sigma}_1\cdot\nabla_1\right]\Psi(1,2,3) \tag{2.1a}$$

$$0=\left[\frac{1}{ic}\frac{\partial}{\partial t_2}+\frac{1}{i}\tau_{2\zeta}\boldsymbol{\sigma}_2\cdot\nabla_2\right]\Psi(1,2,3) \tag{2.1b}$$

$$0=\left[\frac{1}{ic}\frac{\partial}{\partial t_3}+\frac{1}{i}\tau_{3\zeta}\boldsymbol{\sigma}_3\cdot\nabla_3\right]\Psi(1,2,3) \tag{2.1c}$$

which correspond to equations (7.1) of the previous article (Clapp, 1980).

Centroid and relative coordinates are introduced through

$$\begin{aligned}
&\mathbf{R}=(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)/3, \qquad T=(t_1+t_2+t_3)/3\\
&\mathbf{r}=(2\mathbf{r}_1-\mathbf{r}_2-\mathbf{r}_3)/2, \qquad t_r=(2t_1-t_2-t_3)/2\\
&\boldsymbol{\rho}=(\mathbf{r}_2-\mathbf{r}_3), \qquad t_\rho=(t_2-t_3)
\end{aligned} \tag{2.2}$$

We will also need

$$\begin{aligned}
&\frac{\partial}{\partial T}=\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2}+\frac{\partial}{\partial t_3}\\
&\frac{\partial}{\partial t_r}=\frac{2}{3}\frac{\partial}{\partial t_1}-\frac{1}{3}\frac{\partial}{\partial t_2}-\frac{1}{3}\frac{\partial}{\partial t_3}\\
&\frac{\partial}{\partial t_\rho}=\frac{1}{2}\frac{\partial}{\partial t_2}-\frac{1}{2}\frac{\partial}{\partial t_3}
\end{aligned} \tag{2.3}$$

as well as

$$\nabla_1 = \tfrac{1}{3}\nabla_R + \nabla_r$$
$$\nabla_2 = \tfrac{1}{3}\nabla_R - \tfrac{1}{2}\nabla_r + \nabla_\rho \tag{2.4}$$
$$\nabla_3 = \tfrac{1}{3}\nabla_R - \tfrac{1}{2}\nabla_r - \nabla_\rho$$

We will have considerable use for the cyclic representation of the three-item permutation group. Specifically, we will need to use the two complex cube roots of unity, which are

$$\omega = -1/2 + i3^{1/2}/2 \tag{2.5a}$$

$$\omega^2 = -1/2 - i3^{1/2}/2 \tag{2.5b}$$

Using these, we can define cyclic internal gradients by

$$\nabla^+ = \nabla_1 + \omega\nabla_2 + \omega^2\nabla_3 = \tfrac{3}{2}\nabla_r + i3^{1/2}\nabla_\rho \tag{2.6a}$$

$$\nabla^- = \nabla_1 + \omega^2\nabla_2 + \omega\nabla_3 = \tfrac{3}{2}\nabla_r - i3^{1/2}\nabla_\rho \tag{2.6b}$$

Similar cyclic time derivatives can be defined by

$$\frac{\partial^+}{\partial t} = \frac{\partial}{\partial t_1} + \omega\frac{\partial}{\partial t_2} + \omega^2\frac{\partial}{\partial t_3} = \frac{3}{2}\frac{\partial}{\partial t_r} + i3^{1/2}\frac{\partial}{\partial t_\rho} \tag{2.7a}$$

$$\frac{\partial^-}{\partial t} = \frac{\partial}{\partial t_1} + \omega^2\frac{\partial}{\partial t_2} + \omega\frac{\partial}{\partial t_3} = \frac{3}{2}\frac{\partial}{\partial t_r} - i3^{1/2}\frac{\partial}{\partial t_\rho} \tag{2.7b}$$

The wave equations (2.1) can now be replaced by

$$\begin{aligned} 0 = \Big[& \frac{1}{ic}\frac{\partial}{\partial T} + \frac{1}{3i}(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \tau_{2\zeta}\boldsymbol{\sigma}_2 + \tau_{3\zeta}\boldsymbol{\sigma}_3)\cdot\nabla_R \\ & + \frac{1}{3i}\left(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \omega^2\tau_{2\zeta}\boldsymbol{\sigma}_2 + \omega\tau_{3\zeta}\boldsymbol{\sigma}_3\right)\cdot\nabla^+ \\ & + \frac{1}{3i}\left(\tau_{1\zeta}\boldsymbol{\sigma}_1 + \omega\tau_{2\zeta}\boldsymbol{\sigma}_2 + \omega^2\tau_{3\zeta}\boldsymbol{\sigma}_3\right)\cdot\nabla^-\Big]\Psi(1,2,3) \end{aligned} \tag{2.8a}$$

$$0=\left[\frac{1}{ic}\frac{\partial^+}{\partial t}+\frac{1}{3i}(\tau_{1\zeta}\boldsymbol{\sigma}_1+\tau_{2\zeta}\boldsymbol{\sigma}_2+\tau_{3\zeta}\boldsymbol{\sigma}_3)\cdot\boldsymbol{\nabla}^+\right.$$

$$+\frac{1}{3i}\left(\tau_{1\zeta}\boldsymbol{\sigma}_1+\omega^2\tau_{2\zeta}\boldsymbol{\sigma}_2+\omega\tau_{3\zeta}\boldsymbol{\sigma}_3\right)\cdot\boldsymbol{\nabla}^-$$

$$\left.+\frac{1}{3i}\left(\tau_{1\zeta}\boldsymbol{\sigma}_1+\omega\tau_{2\zeta}\boldsymbol{\sigma}_2+\omega^2\tau_{3\zeta}\boldsymbol{\sigma}_3\right)\cdot\boldsymbol{\nabla}_R\right]\Psi(1,2,3) \tag{2.8b}$$

$$0=\left[\frac{1}{ic}\frac{\partial^-}{\partial t}+\frac{1}{3i}(\tau_{1\zeta}\boldsymbol{\sigma}_1+\tau_{2\zeta}\boldsymbol{\sigma}_2+\tau_{3\zeta}\boldsymbol{\sigma}_3)\cdot\boldsymbol{\nabla}^-\right.$$

$$+\frac{1}{3i}\left(\tau_{1\zeta}\boldsymbol{\sigma}_1+\omega^2\tau_{2\zeta}\boldsymbol{\sigma}_2+\omega\tau_{3\zeta}\boldsymbol{\sigma}_3\right)\cdot\boldsymbol{\nabla}_R$$

$$\left.+\frac{1}{3i}\left(\tau_{1\zeta}\boldsymbol{\sigma}_1+\omega\tau_{2\zeta}\boldsymbol{\sigma}_2+\omega^2\tau_{3\zeta}\boldsymbol{\sigma}_3\right)\cdot\boldsymbol{\nabla}^+\right]\Psi(1,2,3) \tag{2.8c}$$

We will assume now that the centroid motion has the form of a plane wave, which we can factor out through the substitution

$$\Psi(1,2,3)=\Phi(1,2,3)\exp(i\kappa\mathbf{k}\cdot\mathbf{R}-i\kappa wcT) \tag{2.9}$$

The reduced wave function $\Phi(1,2,3)$ then satisfies three wave equations similar to (2.8), but with the replacements

$$\frac{1}{ic}\frac{\partial}{\partial T}\to-\kappa w,\qquad \frac{1}{3i}\boldsymbol{\nabla}_R\to\frac{1}{3}\kappa\mathbf{k} \tag{2.10}$$

We will require that the trilocal wave function satisfy a phase-space boundary condition, of the form given earlier in equation (7.9c) of Clapp (1980). In terms of the differential operators (2.4) and (2.6) above, this boundary condition can be written as

$$-3\kappa^2=\nabla_1{}^2+\nabla_2{}^2+\nabla_3{}^2-\tfrac{1}{3}\nabla_R{}^2$$

$$=\tfrac{3}{2}\nabla_r{}^2+2\nabla_\rho{}^2=\tfrac{2}{3}(\boldsymbol{\nabla}^+\cdot\boldsymbol{\nabla}^-) \tag{2.11}$$

3. AUXILIARY CONDITIONS

In the analysis of the bilocal photon structure (Clapp, 1980), we relied strongly on an operator Q_λ which commuted with the Hamiltonian and could therefore be separately specified in the course of sorting out the solutions to the wave equations. There are similar commuting operators in the trilocal analysis.

First we can examine $(\tau_{1\zeta}\boldsymbol{\sigma}_1\cdot\nabla_1)$, which appears as a term in the Hamiltonians and commutes with each. Its square, $\nabla_1{}^2$, will also commute. Similarly will $\nabla_2{}^2$ and $\nabla_3{}^2$. Comparison of these three squares shows that they can be written as

$$\nabla_1{}^2=(S_R+S^-+S^+)/9 \tag{3.1a}$$

$$\nabla_2{}^2=(S_R+\omega S^-+\omega^2S^+)/9 \tag{3.1b}$$

$$\nabla_3{}^2=(S_R+\omega^2S^-+\omega S^+)/9 \tag{3.1c}$$

where

$$S_R=\nabla_R{}^2+2\nabla^+\cdot\nabla^-=-\kappa^2(k^2+9) \tag{3.2a}$$

in which we have used (2.10) and (2.11), and where

$$S^-=(\nabla^+)^2+2\nabla_R\cdot\nabla^- \tag{3.2b}$$

$$S^+=(\nabla^-)^2+2\nabla_R\cdot\nabla^+ \tag{3.2c}$$

The operators S^- and S^+ are not individually suitable candidates for conserved operators of the trilocal system, since they are modified by permutations among the three quanta. [S_R is not so modified, and is given an eigenvalue, as shown in (3.2a).] However, the product of S^- and S^+ is invariant under permutations, and can be given an eigenvalue as defined by

$$S^-S^+=(81/4)\kappa^4\lambda^2 \tag{3.3}$$

There is another combination of S^- and S^+ which is similarly invariant under permutations, and can be given an eigenvalue by

$$(S^-)^3+(S^+)^3=(729/4)\kappa^6\lambda^3\cos(3\mu) \tag{3.4}$$

in which the parameter λ is the same one that appears in (3.3), and the parameter μ is the independent parameter introduced by (3.4). Neither parameter has been limited, up to this point in the analysis, but both will be limited by what follows.

4. SECULAR EQUATION

In the bilocal photon analysis, a secular equation, equation (14.8) of Clapp (1980), was obtained from an explicit secular determinant. An equivalent secular equation could have been obtained through direct operations upon the bilocal Hamiltonian. For the trilocal analysis, it will be convenient to use the direct procedure.

From (2.8a) and (2.10) we can write the operator equation

$$(i\kappa w)=\tau_{1\zeta}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1+\tau_{2\zeta}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_2+\tau_{3\zeta}\boldsymbol{\sigma}_3\cdot\boldsymbol{\nabla}_3 \tag{4.1}$$

Examining the first few powers of this operator, we can obtain the identity

$$\begin{aligned}8(i\kappa w)(\tau_{1\zeta}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}_1)(\tau_{2\zeta}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}_2)(\tau_{3\zeta}\boldsymbol{\sigma}_3\cdot\boldsymbol{\nabla}_3)&=(i\kappa w)^4-2(i\kappa w)^2(\nabla_1{}^2+\nabla_2{}^2+\nabla_3{}^2)\\&\quad+(\nabla_1{}^2+\nabla_2{}^2+\nabla_3{}^2)^2\\&\quad-4(\nabla_1{}^2\nabla_2{}^2+\nabla_2{}^2\nabla_3{}^2+\nabla_3{}^2\nabla_1{}^2)\end{aligned} \tag{4.2}$$

Squaring both sides, and substituting from (3.1)–(3.4), we obtain the secular equation

$$\begin{aligned}0=w^8-\frac{4}{3}w^6(k^2+9)+w^4\left[\frac{10}{27}(k^2+9)^2+6\lambda^2\right]\\+w^2\left[-\frac{28}{729}(k^2+9)^3+\frac{4}{3}(k^2+9)\lambda^2+16\lambda^3\cos(3\mu)\right]\\+\left[\frac{1}{729}(k^2+9)^4-\frac{2}{9}(k^2+9)^2\lambda^2+9\lambda^4\right]\end{aligned} \tag{4.3}$$

We can anticipate that (4.3) is a secular determinant that might appear later in an explicit algebraic solution of a matrix version of the wave equation (2.8a).

If the trilocal structure is to represent an elementary particle with rest mass m (in units of κ), then we will require that w^2 and k^2 satisfy

$$0=w^2-k^2-m^2 \tag{4.4}$$

and (4.4) must accordingly be a factor of (4.3), if (4.3) is to describe a structure which moves as a particle. This relativistic requirement cannot be met if λ and μ are arbitrary. If we substitute (k^2+m^2) for w^2 in (4.3), then the result should be zero for all choices of the squared momentum k^2. From this we can derive the following conditions on λ and μ:

$$\lambda^2 = (9 - m^2)(9 + 7m^2 + 8k^2)/81 \tag{4.5a}$$

$$\lambda^3 \cos(3\mu) = (9 - m^2)^2(9 - 25m^2 - 24k^2)/729 \tag{4.5b}$$

There are three values of $\lambda \cos\mu$ that are consistent with (4.5). These are

$$\lambda \cos\mu = (9 - m^2)/9 \tag{4.6a}$$

$$\lambda \cos\mu' = -(9 - m^2)/18 - (w6^{1/2}/9)(9 - m^2)^{1/2} \tag{4.6b}$$

$$\lambda \cos\mu'' = -(9 - m^2)/18 + (w6^{1/2}/9)(9 - m^2)^{1/2} \tag{4.6c}$$

where we have used

$$w = (k^2 + m^2)^{1/2} \tag{4.7}$$

If we fix the algebraic sign of $\lambda^3 \sin(3\mu)$ through

$$\lambda^3 \sin(3\mu) = w2^{3/2}(9 - m^2)^{3/2}(27 - 11m^2 - 8k^2)/729 \tag{4.8}$$

which is consistent with (4.5), we find

$$\lambda \sin\mu = (w2^{3/2}/9)(9 - m^2)^{1/2} \tag{4.9a}$$

$$\lambda \sin\mu' = -(w2^{1/2}/9)(9 - m^2)^{1/2} + 3^{1/2}(9 - m^2)/18 \tag{4.9b}$$

$$\lambda \sin\mu'' = -(w2^{1/2}/9)(9 - m^2)^{1/2} - 3^{1/2}(9 - m^2)/18 \tag{4.9c}$$

Figure 1 is a plot of $\cos(3\mu)$ against m, with $k^2 = 0$ and with the assumption that λ is positive. The minimum in the curve, where $\cos(3\mu)$ reaches -1, comes at a mass value of

$$9/33^{1/2} = 1.5666989 \tag{4.10}$$

In Figure 2 the rest-system curve of Figure 1 is repeated, together with similar curves in which the momentum is nonzero. All of the curves

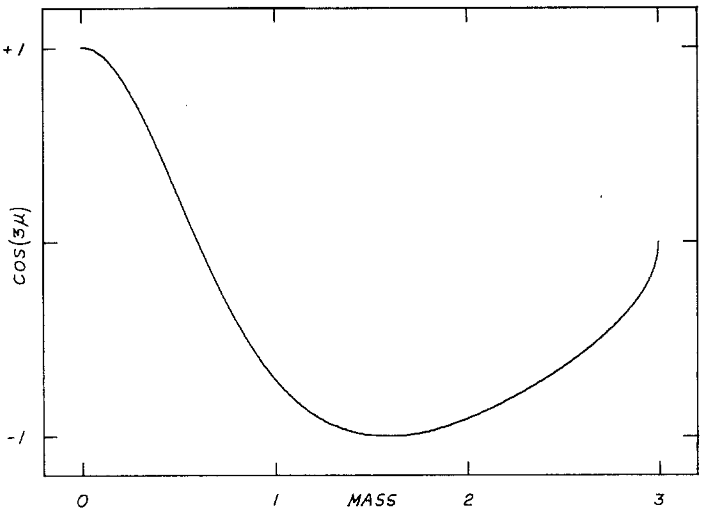

Fig. 1. Plot of $\cos(3\mu)$ as a function of m, for $k^2=0$.

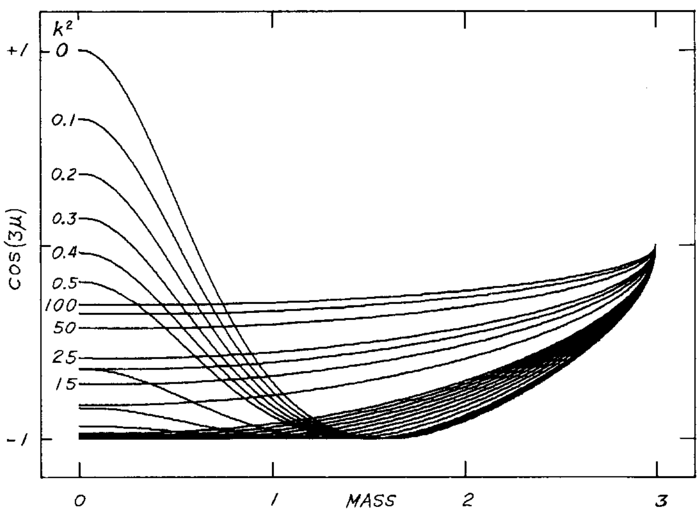

Fig. 2. Plot of $\cos(3\mu)$ as a function of m, for various values of k^2. These values, some of which are indicated on the figure, are 0, 0.1, 0.2, 0.3, 0.4, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 4.5, 5, 10, 15, 20, 25, 50, 75, 100.

approach $\cos(3\mu)=0$ as m approaches 3. This value of m is the upper limit for the mass (in units of κ) for the three-quantum structure. There is a similar upper limit at $m=N$ for the N-quantum structure. Above this limit the N quanta could no longer float on the surface of the vacuum sea.

For small momenta, the curves of Figure 2 are all tangent to the line at $\cos(3\mu)=-1$. At no time does any one of these curves move above or below the region from -1 to $+1$, and this is one justification for the trigonometric way in which the parameter μ was introduced into equation (3.4).

5. SUMMARY

The secular equation (4.3) has been developed from the centroid-time wave equation (2.8a) for the three-quantum system. From this secular equation, together with the requirement that the structure move as a particle subject to the relativistic equation (4.4), the two auxiliary conditions (4.5) have been derived.

As we will see in later articles, the solution to (2.8a) is an infinite expansion, but the conditions (4.5) permit us to replace the coefficients of high-order terms by multiples of a small, finite set of low-order coefficients. The wave equations (2.8) then take the form of a set of linear homogeneous equations in this small set of coefficients. Solutions are found only for certain discrete values of the rest mass m.

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